# EQUILIBRIUM OF A CRACK IN A POROUS MEDIUM WITH INJECTION OF THE FILTERING LIQUID

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In connection with the exploitation of petroleum deposits, the article discusses the equilibrium of a porous medium with a crack under conditions of plane deformation, with the steady-state filtration of a liquid injected into the porous medium through a crack. It is assumed that the crack, which has initial zero dimensions, can become wider and longer with a rise in the pressure. The displacement of the sides of the crack is determined on the basis of the theory of elasticity, taking account of the deformation properties of a saturated porous medium. The stress and the displacement are expressed in terms of two analytical Muskhelishvili functions and the complex filtration potential. A change in the volume of the porous medium leads to a discontinuity of the displacements at the feed contour, and to distortion in the filtration region. For a circular stratum, the dimensions of the crack and the mass flow rate of the liquid are determined in the first approximation. The region of values of the pressure in which there exists a stable equilibrium state of the open crack and a steady-state flow of the liquid is found.

1. With the investigation of filtration in a porous medium with a crack, in the case of a moderate pressure of the liquid the deformation of the crack is generally neglected [1, 2]. In [2] the deformation was partially taken into consideration with unsteady-state filtration, when the width of the crack is small compared to its initial value.

With an increse in the pressure of the saturating liquid, the dimensions of the crack can increase considerably. Such conditions exist with the hydraulic fracture of an oil-bearing stratum, and in the case of the flooding of a stratum at a pressure greater than the well pressure [3, 4].

The dependence between the deformations and the stresses in a porous medium can be represented in the form [5]

$$\varepsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma\delta_{ij}\right) + \beta p\delta_{ij} \tag{1.1}$$
$$i, j = 1, 2, 3; \quad \sigma = \sigma_{ij}\delta_{ij}; \quad \delta_{ij} = 0, \quad i \neq j; \quad \delta_{ij} = 1, \quad i = j$$

where $\mu = E/2(1+\nu)$, E, $\nu$ are the shear modulus, the elastic modulus, and the Poisson coefficient of the skeleton of the porous medium (the dry rock); $\beta = \beta_2 - \beta_1$; $\beta_1$ is the coefficient of linear compressibility of the solid phase (the grains making up the porous medium); $\beta_2 = (1-2\nu)/E$ is the coefficient of linear compressibility of the pressure of the porous medium;

The coefficient $\beta$ depends on the degree of cementing of the rock $\varepsilon = \beta_1/\beta_2$ and determines the change in the volume of the porous medium, as a result of a change in the pressure of the liquid, with a constant tensor of the total stresses $\sigma_{ij}$.

In the limiting case of ideally cemented rocks $\varepsilon \to 1$, we have $\beta \to 0$. The deformation of such rocks is determined only by the total stresses. We note that, with $\beta \to 0$, the porosity $m \to 0$.

In another limiting case of soft soils, when $\varepsilon \to 0$, the deformation is completely determined by the effective stresses $\sigma_{ij}^e = \sigma_{ij} - p\delta_{ij}$.

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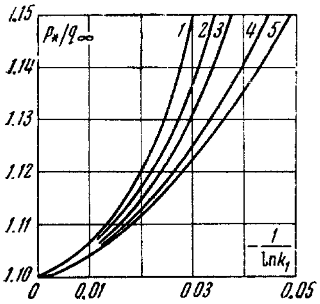

Fig. 1

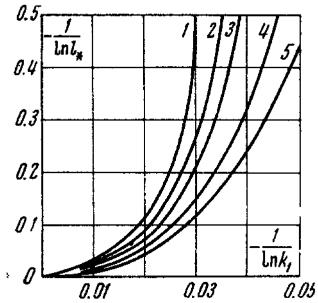

Fig. 2

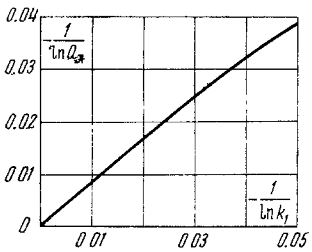

Fig. 3

Using (1.1), the stresses and deformations in a porous medium under conditions of plane deformation of the stratum and plane steady-state flow of the liquid can be expressed in terms of two analytical functions $\varphi(z)$, $\psi(z)$ and the complex filtration potential $w(z)$ $z = x + iy$; the rectangular coordinates x, y are introduced in the plane of the deformation.

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\,\varphi', \quad \sigma_y - \sigma_x + 2i\,\tau_{xy} = 2\,(\bar{z}\varphi'' + \psi')$$
$$2\mu\,(v_x + iv_y) = (3 - 4\nu)\,\varphi - z\overline{\varphi'} - \overline{\psi} + 2\mu\,(1 + \nu)\,\beta\chi \qquad (1.2)$$
$$\chi(z) = \frac{\eta}{k}\int w(z)\,dz$$

where k is the permeability of the porous medium; $\eta$ is the viscosity of the liquid.

Formulas (1.2) can be obtained by a method used with steady-state thermal action [6].

2. Let us consider a homogeneous porous stratum with a circular feed contour, at whose center there is a borehole and a symmetrical vertical crack, directed along the x axis, passing through it. We shall assume that the role of the borehole reduces to a linear source, from which the liquid enters the crack. At the feed contour, the pressure of the liquid is equal to $p_k$. Outside the feed contour, the pressure is constant or there is no liquid.

A change in the volume of the porous medium in the filtration region leads to a discontinuity in the displacements at the feed contour. Therefore, the problem under consideration reduces to a problem in the theory of elasticity for tightly constituted bodies.

$$\varphi_1 + (\omega\,/\,\overline{\omega}')\,\overline{\varphi}_1' + \overline{\psi}_1 = f, \qquad \mathrm{Re}\,w = \frac{k}{\eta}p, \quad \zeta \in \gamma_0 \qquad (2.1)$$

$$\varphi_1 + \frac{\omega}{\overline{\omega}'}\,\overline{\varphi}_1' + \overline{\psi}_1 = \varphi_2 + \frac{\omega}{\overline{\omega}'}\,\overline{\varphi}_2' + \overline{\psi}_2, \quad \mathrm{Re}\,w = \frac{k}{\eta}p_k, \quad \zeta \in \gamma_k \qquad (2.2)$$
$$(3 - 4\nu)\,\varphi_1 - \frac{\omega}{\overline{\omega}'}\,\overline{\varphi}_1' - \overline{\psi}_1 + 2\mu\,(1 + \nu)\,\beta P = (3 - 4\nu)\,\varphi_2 - \frac{\omega}{\overline{\omega}'}\,\overline{\varphi}_2' - \overline{\psi}_2, \quad \zeta \in \gamma_k$$

$$P(\zeta) = \int\left[\frac{\eta}{k}w(\zeta) - p_k\right]\omega'(\zeta)\,d\zeta \qquad (2.3)$$

The transform $z = \omega(\zeta)$ brings the filtration region to a circular concentric annulus in the plane $\zeta = \rho e^{i\theta}$. The annulus is bounded by the circles $\gamma_0$ and $\gamma_k$, having the radii 1 and $\rho_k$.

Assuming that the crack is small compared with the dimensions of the stratum, we can take $\omega = \frac{1}{2}l(\zeta + 1/\zeta)$, where $2l$ is the length of the crack. Then the circle with the radius $\rho_k = 2R_k/l$, where $R_k$ is the radius of the stratum, corresponds to an elliptical contour in the plane of z, close to the feed contour. The deviation of the elliptical contour from the circular contour of the stratum with $l < 0.2R_k$ does not exceed $0.01\,R_k$.

For the principal vector of the forces applid to the contour of the crack we have

$$f = -\frac{l}{2}\int p(\sigma)\left(\sigma - \frac{1}{\sigma}\right)\frac{d\sigma}{\sigma}, \qquad \sigma = e^{i\theta} \qquad (2.4)$$

where $p(\sigma)$ is the pressure at the contour of the crack. The tangential stresses due to viscous friction have only a negligible effect on the deformation of the crack.

The volumetric changes in the stratum brought about by injection of the liquid correspond to the distortion. As a result of the symmetry of the problem with respect to the x and y axes, there is only a rotational component of the distortion. Therefore,

$$\varphi_1 = a\frac{\eta Q}{2\pi kh}\,\frac{l}{4}\left(\zeta + \frac{1}{\zeta}\right)\ln\zeta + \varphi_{10}, \quad \psi_1 = \psi_{10}, \quad a = \mu\frac{1+\nu}{1-\nu}\beta \qquad (2.5)$$

where Q is the mass flow rate of the liquid; h is the thickness of the stratum; $\varphi_{10}$ and $\psi_{10}$ are holomorphic between $\gamma_0$ and $\gamma_k$.

The functions $\varphi_2$ and $\psi_2$ have the form

$$\varphi_2 = -{}^1/_4\, lq_\infty\zeta + \varphi_{20}, \quad \psi_2 = \psi_{20} \tag{2.6}$$

where $q_\infty$ is the lateral stratum pressure; $\varphi_{20}$ and $\psi_{20}$ are holomorphic outside of $\gamma_k$.

The constants in $f$, P, and $\varphi_{20}$ can be assumed equal to zero. Then the constant in $\varphi_{20}$ is found with solution of the problem (2.1)-(2.3).

For equilibrium of the crack, the stresses at its ends must be finite,

$$\operatorname{Re} \varphi_1'(\pm 1) = 0 \tag{2.7}$$

The boundary condition connecting the stress and the displacement at the contour of the crack is obtained from the equations of motion and the conditions of the conservation of the mass of the liquid in the crack.

With plane laminar motion, for the volumetric mass flow rate of the liquid through a cross section of the crack we have [1]

$$q = -h\frac{2d}{\eta}\left(\frac{d^2}{3} + k\right)\frac{\partial p}{\partial x} \tag{2.8}$$

where $2d = v_y(x, +0) - v_y(x, -0)$ is the opening of the crack.

The condition for conservation of the mass of an incompressible liquid is represented in the form

$$dq / dx + h\,(u_y^+ - u_y^-) - Q\delta\,(x) = 0 \tag{2.9}$$

where $u_y^+ = u_y(x, +0)$, $u_y^-(x, -0)$ are the filtration rates through the sides of the crack; $\delta(x)$ is a delta function.

Integrating (2.9) after substitution of (2.8) and going over to variables in the plane $\zeta$, we can obtain the boundary condition at the contour of the crack in the form

$$\frac{4v_y}{\eta l \sin\theta}\left(\frac{v_y^2}{3} + k\right)\frac{\partial p}{\partial \theta} + \frac{4k}{\eta} r(\theta)\int_0^\theta r(\theta)\frac{\partial p}{\partial \rho}\,d\theta + \frac{Q}{h}\left[r(\theta) - r(2\theta)\right] = 0 \tag{2.10}$$

where the values of the derivatives are taken with respect to $\gamma_0$; $r(\theta) = \operatorname{sign}(\sin\theta)$ is a Rademacher function.

3. We represent the pressure in the crack in the form

$$p = p_0 + \sum_{n=2}^{\infty} p_n \cos n\theta \tag{3.1}$$

As a result of the symmetry of the problem with odd values of n, we have $p_n = 0$.

Taking account that $\rho_k \gg 1$, we further neglect terms of the order $1/\rho_k$ and higher.

For the filtration potential, the pressure, and the mass flow rate of the liquid we have

$$w = \frac{k}{\eta}\left[p_k + (p_0 - p_k)\left(1 - \frac{\ln\zeta}{\ln\rho_k}\right) + \sum_{n=2}^{\infty}\frac{p_n}{\zeta^n}\right] \tag{3.2}$$

$$p = p_k + (p_0 - p_k)\left(1 - \frac{\ln\rho}{\ln\rho_k}\right) + \sum_{n=2}^{\infty}\frac{p_n}{\rho^n}\cos n\theta \tag{3.3}$$

$$Q = \frac{2\pi kh}{\eta}\frac{p_0 - p_k}{\ln\rho_k} \tag{3.4}$$

Taking account of (3.2), we find

$$P = \frac{p_0 - p_k}{\ln\rho_k}\left[\zeta\omega'(\zeta) + \omega(\zeta)\ln\rho_k - \omega(\zeta)\ln\zeta\right] + P_0$$

$$P_0 = \frac{l}{2}\sum_{n=2}^{\infty} p_n\left(\frac{\zeta^{-n-1}}{n+1} - \frac{\zeta^{-n+1}}{n-1}\right) \tag{3.5}$$

We bring the problem (2.1)-(2.3) to a boundary-value problem for a crack in an infinite region with continuous displacements.

From (2.2) and (2.3) we find

$$\varphi^+ - \varphi^- = -q_\infty \frac{l}{4} t - \frac{a}{2}(p_0 - p_k)\left[\omega(t) - \frac{t\omega'(t)}{\ln \rho_k}\right] - \frac{a}{2} P_0(t)$$
$$\psi^+ - \psi^- = \frac{a}{2}\frac{p_0 - p_k}{\ln \rho_k}\overline{\omega'(t)}\,t + \frac{a}{2} P_0\left(\frac{\rho_k^2}{t}\right) + \frac{a}{2}\frac{\overline{\omega}}{\omega'} P_0'(t)$$
$$t = \rho_k e^{i\theta}, \quad \varphi^+(t) = \varphi_{10}(t), \quad \varphi^-(t) = \varphi_{20}(t)$$
$$\psi^+(t) = \psi_{10}(t), \quad \psi^-(t) = \psi_{20}(t) \tag{3.6}$$

Introducing the piecewise holomorphic functions

$$\varphi_* = \frac{1}{2\pi i}\int_{\gamma_k}(\varphi^+ - \varphi^-)\frac{dt}{t-\zeta}, \quad \psi_* = \frac{1}{2\pi i}\int_{\gamma_k}(\psi^+ - \psi^-)\frac{dt}{t-\zeta}$$

we represent $\varphi_{10}$ and $\psi_{10}$ in the form

$$\varphi_{10} = \varphi_0 + \varphi_*, \quad \psi_{10} = \psi_0 + \psi_* \tag{3.7}$$

where $\varphi_0$ and $\psi_0$ are holomorphic outside of $\gamma_0$.

Substituting (3.7) into (2.1), we obtain

$$\varphi_0 + \frac{\omega}{\overline{\omega}'}\overline{\varphi}_0' + \overline{\psi}_0 = f_0, \quad f_0 = f - \varphi_* - \frac{\omega}{\overline{\omega}'}\overline{\varphi}_*' - \overline{\psi}_* \tag{3.8}$$

The solution of the problem (3.8) is known [6]. As a result, we find

$$\varphi_1 = -\frac{l}{4}q_\infty\left(\zeta - \frac{1}{\zeta}\right) - \frac{l}{2}p_0\frac{1}{\zeta} - \frac{l}{4}\sum_{n=2}^{\infty}{}' p_n\left(\frac{\zeta^{-n-1}}{n+1} - \frac{\zeta^{-n+1}}{n-1}\right) + a\frac{l}{4}\frac{p_0 - p_k}{\ln \rho_k}\left(\zeta + \frac{1}{\zeta}\right)\ln\zeta - a\frac{l}{4}(p_0 - p_k)\left(1 + \frac{1}{\ln \rho_k}\right)\left(\zeta - \frac{1}{\zeta}\right) \tag{3.9}$$

From the condition for equilibrium of the crack (2.7) we have

$$p_0 = (q_\infty - ap_k)\,/\,(1 - a) \tag{3.10}$$

For the displacement of the points of the contour of the crack we find

$$v_y = l\frac{1 - \nu^2}{E}(1 - a)\sum_{n=2}^{\infty} p_n\left(\frac{\sin(n+1)\theta}{n+1} - \frac{\sin(n-1)\theta}{n-1}\right) \tag{3.11}$$

It is easily seen that, as in a solid body, the condition of the finite character of the stresses in a saturated porous medium ensures the smoothness of the closing of the crack.

We introduce the units of measurements of the length $R_k$, the stress $E/(1-\nu^2)$, and the mass flow rate of the liquid $\pi h R_k^2 E/\eta\,(1-\nu^2)$. For the dimensionless quantities, we retain the notation of the corresponding dimensional quantities.

Taking account of (3.10), the mass flow rate of the liquid assumes the form

$$Q = \frac{4k_1(q_\infty - p_k)}{(1-a)\ln 4/l^2}, \quad k_1 = \frac{k}{R_k^2} \tag{3.12}$$

We substitute (3.3), (3.11), and (3.12) into (2.10):

$$\frac{1}{\sin\theta}\sum_{n=2}^{\infty} np_n \sin n\theta\left\{\frac{l^2}{3}\left[(1-a)\sum_{n=2}^{\infty} lp_n\left(\frac{\sin(n+1)\theta}{n+1} - \frac{\sin(n-1)\theta}{n-1}\right)\right]^3 + k_1(1-a)\sum_{n=2}^{\infty} lp_n\left(\frac{\sin(n+1)\theta}{n+1} - \right.\right.$$
$$\left.\left. - \frac{\sin(n-1)\theta}{n-1}\right)\right\} + k_1\sum_{n=2}^{\infty} np_n r(\theta)\int_0^\theta r(\theta)\cos n\theta\, d\theta + k_1\frac{2(q_\infty - p_k)}{(1-a)\ln 4/l^2}\left\{r(\theta)\int_0^\theta r(\theta)\, d\theta - \frac{\pi}{2}[r(\theta) - r(2\theta)]\right\} = 0 \tag{3.13}$$

For r we have

$$r(\theta) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sin n\vartheta}{n}$$

Multiplying (3.13) by sin $n\theta$, integrating from 0 to $2\pi$, and then setting n = 2, 4, 6, ..., we can obtain an infinite system of algebraic equations of the fourth order with respect to the coefficients $p_n (n \geq 2)$, into which only the length of the crack enters. (The equations obtained with 1, 3, 5, ..., are satisfied identically).

If the mass flow rate of the liquid is given, the length of the crack is determined from (3.12).

When the mass flow rate of the liquid is not known, but the end-face pressure of the borehole is known, $p_c$ (with x = 0), the system obtained is closed by the equation

$$p_c = p_0 + \sum_{n=2}^{\infty} (-1)^{n/2} p_n \tag{3.14}$$

4. The character of the equilibrium of the crack is determined by the form of the dependence of $p_c$ on $l$. The function $p_c(l)$ varies nonmonotonically. In actuality, with $l \to 0$ and $l \to \infty$, from (3.13) we obtain a system of homogeneous equations having the trivial solution $p_n = 0$. Therefore, $p_c(0) = p_0$ and $p_c(\infty) = p_0$, while the function $p_c(l)$ has at least one maximum.

The point $l = 0$, $p_c = p_0$ corresponds to the opening of the crack. Then, with a rise in the value of $l$, $p_c$ increases.

We postulate that the first maximum of $p_c(l)$ is attained at a point $l_* \ll 1$. Then, in the segment $l > l_*$, a decrease in the value of the function $p_c(l)$ corresponds to a growth of the crack with a decrease in the injection pressure. Therefore, the point $l_*$, $p_* = p_c(l_*)$ is the limiting point of a stable equilibrium state and of steady-state flow of the liquid.

Thus, an open crack can exist in equilibrium only with an injection pressure belonging to the region $p_0 < p_c < p_*$.

It can be seen from (3.10) that, except for the case $a = 0$ and $p_k = q_\infty$, the pressure of the opening of the crack always exceeds the stratum pressure. This is due to the fact that, with an increase in the pressure of the saturating liquid, the volume of rock inside the feed contour increases. The far part of the stratum, where the pressure does not vary, prevents the free expansion of the rocks and an additional compressive stress arises in them. The case $a = 0$ corresponds to ideally cemented rocks, not subject to volumetric changes. With $p_k = q_\infty$, the effective stress in the rocks is equal to zero; therefore, for opening of the crack it is sufficient to increase the pressure by an arbitrarily small amount.

The permissible values of $a$ lie within the limits from 0 to 0.5. With $\nu = 0.2$, for soft soils we obtain $a = 0.375$. On the basis of the experiments of Fatt for the rocks of oil-bearing strata, the degree of cementing has a value $\varepsilon = 0.11$–$0.38$ [5]. In these cases, we obtain $a = 0.232$–$0.335$. Consequently, with steady-state filtration under the conditions of oil-bearing strata, the pressure of the opening of a crack always exceeds the stratum pressure by 1.1–1.2 times.

Let us consider the solution of the problem, limiting ourselves to one term in the summation of (3.1),

$$p = p_0 + p_2 \cos 2\theta \tag{4.1}$$

From (3.13) we obtain

$$\frac{7}{27}(1-a)^3 l^2 p_2^4 + \frac{4}{3}(1-a) k_1 p_2^2 - k_1 p_2 - \frac{2k_1 (q_\infty - p_k)}{(1-a)\ln 4/l^2} = 0 \tag{4.2}$$

Since $|p_2| \ll 1$, the second quadratic term in (4.2) can be neglected.

For the limiting values we find

$$l_*^2 \frac{(\ln 4/l_*^2 - 1)^4}{(\ln 4/l_*^2)^6} - \frac{27}{7 \cdot 8} \frac{k_1}{(q_\infty - p_k)^3} = 0 \tag{4.3}$$

$$p_* = p_0 + \frac{2(q_\infty - p_k)}{1-a} \frac{\ln 4/l_*^2 - 1}{(\ln 4/l_*^2)^2} \tag{4.4}$$

With k = 0 we have $l_* = 0$ and $p_* = p_0$. It is obviously true that in an impermeable medium with the injection of a liquid a crack cannot be in equilibrium.

With a rise in the permeability, the limiting pressure, the length of the crack, and the mass flow rate of the liquid increase. With an increase in the stratum pressure, the limiting injection pressure and the length of the crack decrease, while, in the cases calculated, the limiting mass flow rate practically does not change (on Figs. 1-3 the curves 1, 2, 3, 4, and 5 correspond to the values $q_\infty = 5 \cdot 10^{-4}$, $2.5 \cdot 10^{-3}$, $5 \cdot 10^{-3}$, and $5 \cdot 10^{-2}$ with $a = 0.335$ and $p_k/q_\infty = 0.8$).

For cracks of small length, when the condition is satisfied

$$l^2 \, [2 \, (q_\infty - p_k) / \ln 4 / l^2]^3 \ll {}^{27}/_7 k_1 \tag{4.5}$$

the first term in (4.2) can be neglected. Consequently, a predominant part of the liquid is filtered through a small central part of the crack. Then we find

$$l = 2 \exp\left(-\frac{p_0 - p_k}{p_c - p_0}\right) \tag{4.6}$$

In this case, the mass flow rate depends linearly on the injection pressure

$$Q = 2k_1 \, (p_c - p_0) \tag{4.7}$$

The acceptance capacity of a crack $dQ/dp_c$ is $\ln R_k/r_c$ times greater than the acceptance capacity of a borehole of radius $r_c$.

Under the conditions of oil-bearing strata, the region of existence of a stable equilibrium of a crack is not great.

Let $q_\infty = 125$ kg/cm², $p_k = 100$ kg/cm², $R_k = 500$ m, $h = 1$ m, $k = 1$ d, $\eta = 1$ cP, $E = 10^5$ kg/cm², $\nu = 0.2$, $a = 0.335$. Then equilibrium of a crack is possible with injection pressures from 138 to 142 kg/cm². For limiting values, we have $l_* = 1.13$ m; $Q_* = 1.03$ dm³/min; opening in middle of crack 0.192 mm; volume of crack 0.253 dm³. In accordance with (4.6), (4 7), condition (4.5) is satisfied with $l < 0.5$ m.

## LITERATURE CITED


1. I. M. Abdurakhmanov, "Perturbation of a filtration flow by a single crack," Prikl. Matem. Mekhan., 33, No. 5, 871-875 (1969).
2. V. P. Pilatovskii, "Solution of problems in the hydromechanics of microcracks in a thin stratum with the filtration of a homogeneous liquid," Prikl. Matem. Mekhan., 35, No. 3, 532-544 (1971).
3. Yu. P. Zheltov and S. A. Khristianovich, "Hydraulic fracture of an oil-bearing stratum," Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, No. 5, 3-41 (1955).
4. Yu. P. Zheltov, "The formation of vertical cracks in a stratum with a filtering liquid," Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, No. 8, 56-62 (1957).
5. V. I. Nikolaevskii, K. S. Basniev, A. T. Gorbunov, and G. A. Zotov, The Mechanics of Saturated Porous Media [in Russian], Izd. Nedra, Moscow (1970), p. 334.
6. A. I. Lur'e, The Theory of Elasticity [in Russian], Izd. Nauka, Moscow (1970), p. 939.